Review on de Bruijn shapes in 1, 2 and 3 dimensions

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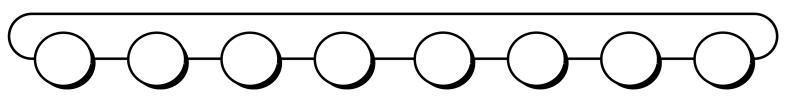
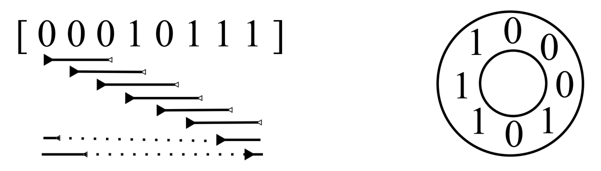
De Bruijn sequence

- A de Bruijn sequence B(k,n) is the shortest k^n-length string composed by all n-*length* substrings available with a k-ary alphabet containing each of those substrings just once, where $k \geq 2$ and $n \geq 1$
- It has 1 dimension and its shape is that of a toroidal array



- Reflection on a sequence gives a new instance, but rotation don't

Instance of B(2,3)

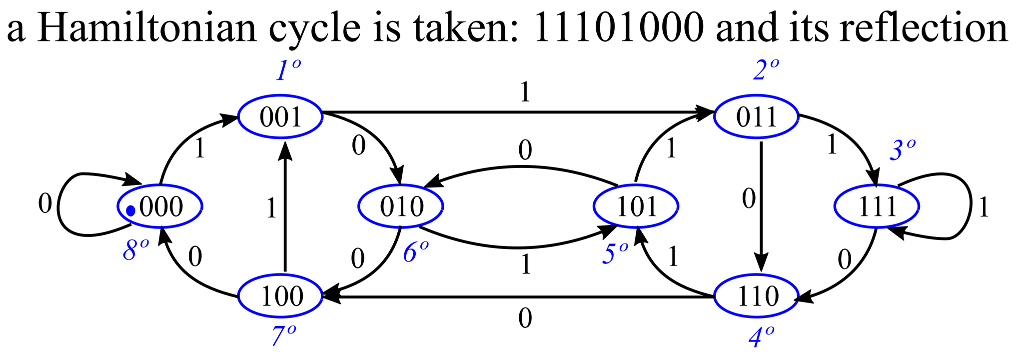


- There are $\frac{(k!)^{k^{n-1}}}{k^n}$ different sequences for given values of k and n
- They allow to precisely spot any particular substring within the string

De Bruijn graph

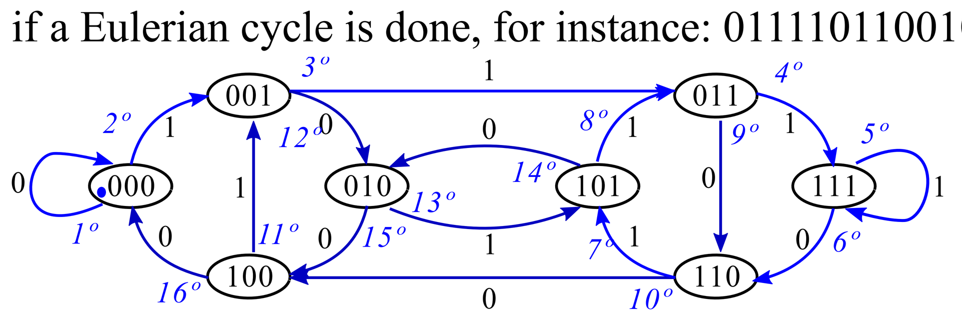
- Directed graph with k^n nodes, allowing to obtain de Bruijn sequences
- $B(k,n)$ if a Hamiltonian cycle is taken: 11101000 and its reflection 00010111

B(2,3)



- $B(k,n+1)$ if a Eulerian cycle is done, for instance: 0111101100101000

B(2,4)



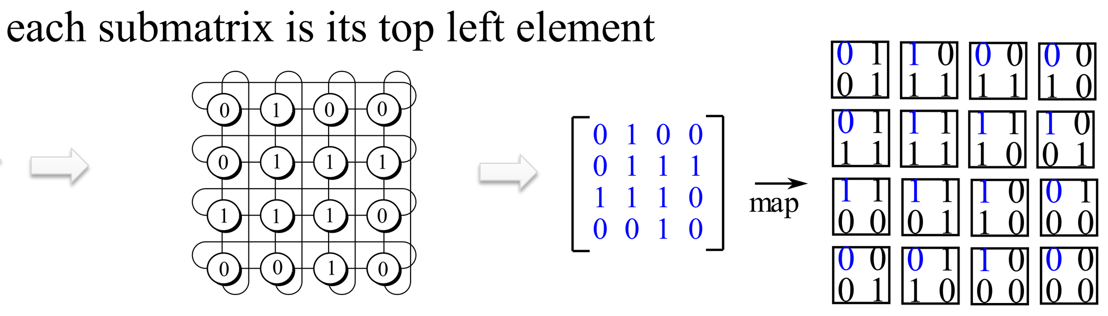
All 16 B(2,4) cycles

0000110100101111	1111001011010000	1111010010110000	0000101101001111
0000100110101111	1111011001010000	1111010110010000	0000101001101111
0000101100111101	1111010011000010	1011110011010000	0100001100101111
0000110101111001	1111001010000110	1001111010110000	0110000101001111

De Bruijn Torus

- Extension of the concept of de Bruijn sequences to 2 dimensions
- A de Bruijn torus $(r_1, r_2; m_1, m_2)_k$ is the smallest $r_1 \times r_2$ matrix with all possible $m_1 \times m_2$ patterns in a k-ary alphabet just once
- Its shape is a toroidal matrix and patterns are toroidal submatrices
- A sufficient condition for a de Bruijn torus to exist:
$$r_1 \times r_2 = k^{m_1 \times m_2}$$
- They allow to exactly spot any particular 2D-pattern within the matrix
- The handle of each submatrix is its top left element

Instance of $(4,4;2,2)_2$



De Bruijn 3D-hypertorus

- Extension of the concept of de Bruijn torus to 3 dimensions
- A de Bruijn 3D-hypertorus $(r_1, r_2, r_3; m_1, m_2, m_3)_k$ is the smallest $r_1 \times r_2 \times r_3$ 3D-hypermatrix with all possible $m_1 \times m_2 \times m_3$ patterns in a k-ary alphabet appearing exactly once
- Its shape is a 3D-hypertoroidal matrix and patterns are 3D-hypert.submatrices
- A sufficient condition for a de Bruijn 3D-hypertorus to exist:
$$r_1 \times r_2 \times r_3 = k^{m_1 \times m_2 \times m_3}$$
- They allow to spot any particular 3D-pattern within the 3D matrix
- The handle of each 3D submatrix is its front top left element
- Each handle gives the distances to the front top left corner of the 3D matrix, which is considered to be (0,0,0), such that layer 0, row 0 and column 0

Some instances of de Bruijn sequences

k	B(k,n)	One particular instance of that $B(k,n)$
$\{0,1\}$	$B(2,2)$	1100
$\{0,1\}$	$B(2,3)$	11101000
$\{0,1\}$	$B(2,4)$	1111010110010000
$\{0,1\}$	$B(2,5)$	11111000110111010100000100101100
$\{0,1\}$	$B(2,6)$	1111110110111010101111001011001101001001110001010001100001000000
$\{0\cdots2\}$	$B(3,2)$	221100201
$\{0\cdots2\}$	$B(3,3)$	222121110001122012020021010
$\{0\cdots3\}$	$B(4,2)$	3322110023130120
$\{0\cdots3\}$	$B(4,3)$	3330300221121222020010111000312032102132013012302310331131332232
$\{0\cdots4\}$	$B(5,2)$	4431323303422414021120100
$\{0\cdots5\}$	$B(6,2)$	554453433524232251413121150403020100
$\{0\cdots6\}$	$B(7,2)$	6654562336424463435325226155051304160314021120100
$\{0\cdots7\}$	$B(8,2)$	7766756557464544736353433726252423227161514131211706050403020100

Wong Algorithm

- An algorithm for binary and k-ary alphabets allowing to achieve de Bruijn sequences
  - A necklace is the lexicographically earliest string within a rotation set

$$f_2(b_1 b_2 \cdots b_n) = \begin{cases} b_2 b_3 \cdots b_n \overline{b_1} & \text{if } b_2 b_3 \cdots b_n b_1 \text{ is a necklace;} \\ b_2 b_3 \cdots b_n b_1 & \text{otherwise.} \end{cases}$$

$$f_k(a_1 a_2 \cdots a_n) = \begin{cases} a_2 a_3 \cdots a_n b & \text{if } a_1 = k-1; \\ a_2 a_3 \cdots a_n (a_1+1) & \text{if } a_1 \neq k-1 \text{ and } a_2 a_3 \cdots a_n (a_1+1) \text{ is a necklace;} \\ a_2 a_3 \cdots a_n a_1 & \text{otherwise.} \end{cases}$$

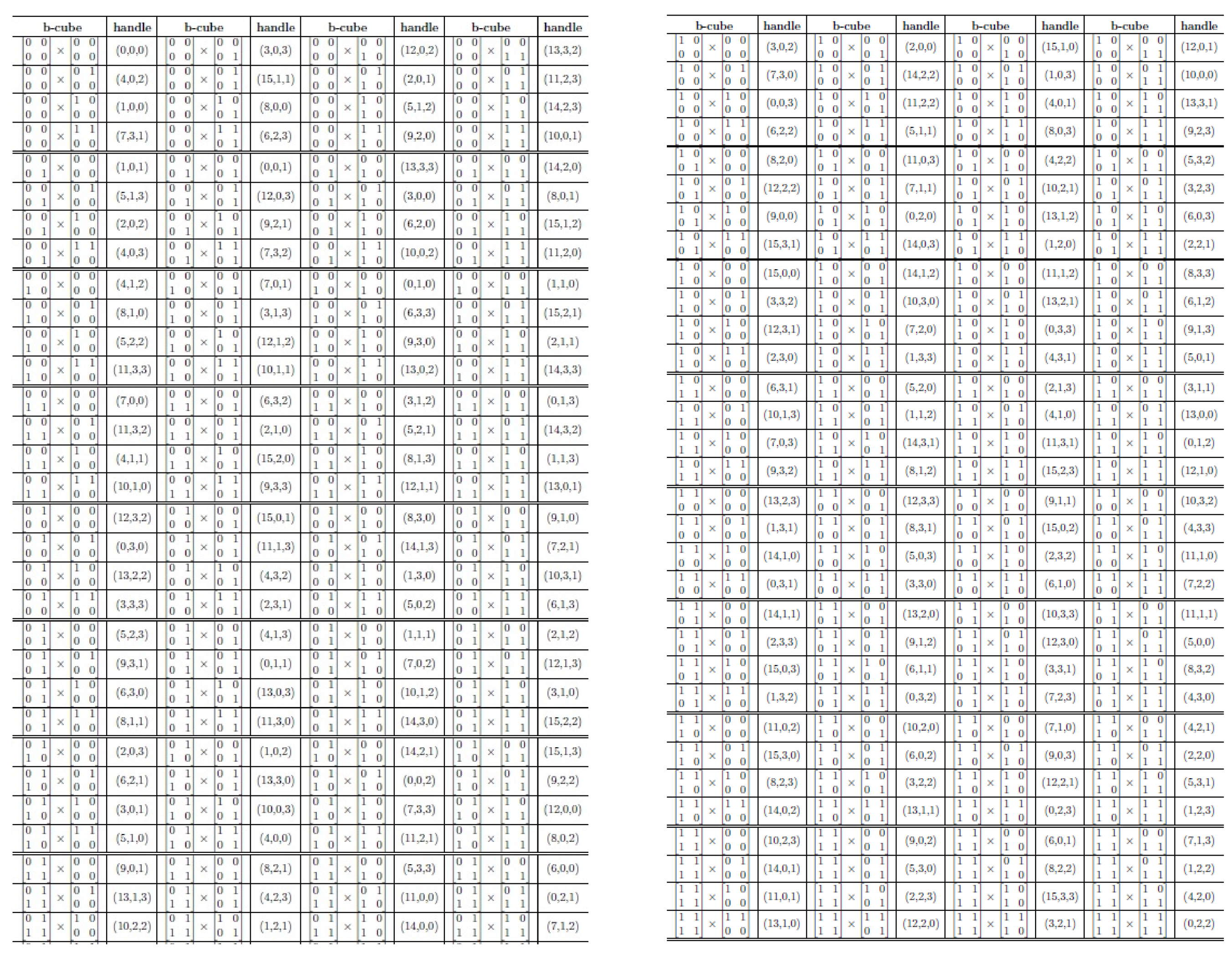
Now: $b_1b_2b_3$	Test: b_2b_31	Necklace?	Next: YES: $b_2b_3\overline{b_1}$ NO: $b_2b_3b_1$
000	001	Yes	001
001	011	Yes	011
011	111	Yes	111
111	111	Yes	110
110	101	No	101
101	011	Yes	010
010	101	No	100
100	001	Yes	000

  - Taking the last digit of each string of the last column, from top downwards, the result is B(2,3): 11101000, which is the same as the one with de Bruijn graph

Some instances of de Bruijn tori

$(16,32;3,3)_2$	$(8,8;3,2)_2$	$(4,16;3,2)_2$	$(9,9;2,2)_3$	$(16,16;2,2)_4$
00000010111001011100101110010111	00000101	0000011101010011	000100010	0010001030203020
00010011111101001101101010000110	00100111	0011001100100110	001221221	0001020301000203
00000010111001011100101110010111	00110110	0011001010111011	111121211	0111011131213121
11101100000010110010010101111001	01000001	1111100100110001	112002002	1011121311101213
00000010111001011100101110010111	11111010		221201021	0010001030203020
00010011111101001101101010000110	11011000		110210120	2021222321202223
00110001110101101111100010100100	11001001		002012102	0111011131213121
10001010011011010100001100011111	10111110		222022202	3031323331303233
11111101000110100011010001101000			220110110	0313031333233323
11101100000010110010010101111001				1011121311101213
10101000010011110110000100111101				0212021232223222
01000110101000011000111111010011				0001020301000203
11111101000110100011010001101000				0313031333233323
11101100000010110010010101111001				2021222321202223
01100100100000111010110111110001				0212021232223222
110111110011100000001011001001010				3031323331303233

Mapping of de Bruijn 3D hypertorus $(16,4,4;2,2,2)_2$



Instance of de Bruijn 3D hypertorus

b−cube $2 \times 2 \times 2$

Instance of $(16,4,4;2,2,2)_2$